Topics in Algebra solution

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Problems in Section 5.7.

1. If $p(x)$ is solvable by radicals over F, prove that we can find a sequence of fields

$$F \subset F_1 = F(w_1) \subset F_2 = F_1(w_2) \subset \cdots \subset F_k = F_{k-1}(w_k),$$

where $w_1^{r_1} \in F$, $w_2^{r_2} \in F, \cdots, w_k^{r_k} \in F_{k-1}$, F_k containing all the roots of $p(x)$, such that F_k is normal over F.

Proof. We claim that

Lemma. If $B = F(w_1, w_2, \cdots, w_k)$ is a finite extension of field F, then there is a finite extension E, that is a splitting field of some polynomial $f(x) \in F[x]$. Such an extension is called a normal closure of B over F.
($\Rightarrow$) Since every w_i is algebraic over F, there exists an irreducible polynomial $p_i(x) \in F[x]$ such that $p_i(w_i) = 0$ for each i. Let $f(x) = \prod_{i=0}^{k} p_i(x)$. Then the splitting field E of $f(x)$ over F is in fact, the normal extension of $f(x)$ over F containing every w_i.

Now we claim that such obtained extension E is a radical extension of F.

Lemma. If B is a radical extension of F, then the extension E is a radical extension of F.
($\Rightarrow$) By Lemma 5.6.3, for each pair of roots w and w' of an irreducible factor $p_i(x)$ of $f(x)$, there exists an automorphism $\sigma \in G(E, F)$ such that $w \mapsto w'$. It follows that $E = F(\sigma(u_1), \sigma(u_2), \cdots \sigma(u_t) : \sigma \in G(E, F))$. If B is a radical extension of F, then

$$F \subset F(u_1) \subset F(u_1, u_2) \subset \cdots \subset F(u_1, u_2, \cdots, u_t) = B.$$

Clearly, $\sigma(B) = F(\sigma(u_1), \sigma(u_2), \cdots, \sigma(u_t))$ is a radical extension for any $\sigma \in G(E, F)$. Let $B_1 = F(\sigma(u_1) : \sigma \in G(E, F))$. If we display $G(E, F) = \{1, \sigma, \tau \cdots\}$, we have

$$F \subset F(u_1) \subset F(u_1, \sigma(u_1)) \subset F(u_1, \sigma(u_1), \tau(u_1)) \subset \cdots \subset B_1$$

which implies that B_1 is a radical extension. Now by induction, suppose we constructed radical chain B_i containing $\{\sigma(u_j) : \sigma \in G(E,F)\}$ for all $j \leq i$. Define

$$B_{i+1} = B_i(\sigma(u_{i+1}) : \sigma \in G(E,F))$$

If $u_{i+1}^m \in F(u_1, u_2 \cdots, u_i)$, then $\tau(u_{i+1})^m \in F(\tau(u_1), \tau(u_2) \cdots, \tau(u_i)) \subset B_i$. It follows that B_{i+1} is a radical extension of F. Since $E = B_t$, E is a radical extension of F.

Now apply the two lemmas; the result is straightforward. □

2. Prove that a subgroup of a solvable group is solvable.

Proof. Refer the Problem 10, Supplementary Problems in Chapter 2. □

3. Prove that S_4 is a solvable group.

Proof. Every group of order less than 60 is solvable. □

4. If G is a group, prove that all $G^{(k)}$ are normal subgroups of G.

Proof. Refer the Problem 13 a), Supplementary Problems in Chapter 2. □

5. If N is a normal subgroup of G prove that N' must also be a normal subgroup of G.

Proof. Let $a, b \in N$. Choose $g \in G$. Denote $gag^{-1} = a'$ and $gbg^{-1} = b'$. Clearly, $(gag^{-1})^{-1} = ga^{-1}g^{-1} = (a')^{-1}$ and similarly, $(gbg^{-1})^{-1} = (b')^{-1}$. Observe that

$$\begin{aligned} g \cdot aba^{-1}b^{-1} \cdot g^{-1} &= ga(g^{-1}g)b(g^{-1}g)a^{-1}(g^{-1}g)b^{-1}(g^{-1}g)g^{-1} \\ &= (gag^{-1})(gbg^{-1})(ga^{-1}g^{-1})(gbg^{-1}) \\ &= a'b'(a')^{-1}(b')^{-1} \end{aligned}$$

which implies that N' is also a normal subgroup of G. □

6. Prove that the alternating group (the group of even permutations in S_n) A_n has no nontrivial normal subgroups for $n \geq 5$.

Proof. Refer the arguments made in Problem 14 and 17, Section 2.10. □