Topics in Algebra solution

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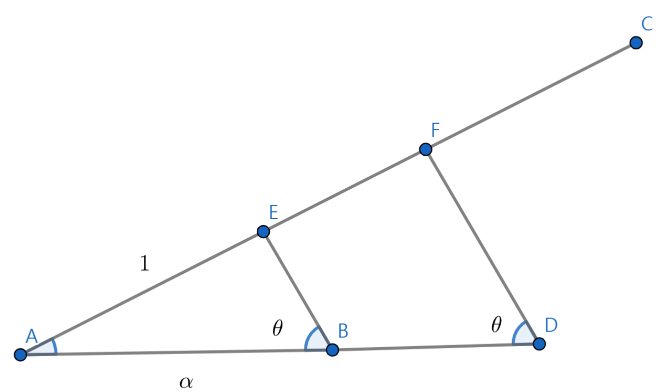
Problems in Section 5.4.

1. Prove that if α, β are constructible, then so are $\alpha \pm \beta$, $\alpha\beta$, α/β (when $\beta \neq 0$) .

Proof. Suppose given that α and β are constructible. Without lossing of generality, we shall assume that $\alpha, \beta > 0$ and $\alpha > \beta$(if necessary). We begin by drawing a circle C_1 on the Euclidean plane with centre O with radius α. Again, draw a circle C_2 of radius β with any point B on C_1 as its centre. Then the straight line which joins O and B intersects C_2 at two points. Denote the nearest intersection point as P_1 and the other one as P_2. Then the line segment $\overline{OP_1}$ has the length $\alpha - \beta$ while the line segment $\overline{OP_2}$ has the length $\alpha + \beta$. Hence, $\alpha \pm \beta$ is contructible.



$Fig.1$

Now we claim that $\alpha\beta$ is construcitible. WLOG, we assume that $\beta > 1$. As in Fig.1, draw a line joining A and D and mark B on the that line so that $\overline{AB}$ is a line segment with length α. Now assume that there is a straight line joining A and C and mark E on the same line so that $\overline{AE}$ has length 1. Let $\overline{F}$ be the line segment of length β. Suppose $\overline{AD}$

has length x. From the congruence property of triangle $\triangle AEB$ and $\triangle AFD$, we have that

$$\frac{|\overline{AE}|}{|\overline{AB}|} = \frac{|\overline{AF}|}{|\overline{AD}|} = \frac{1}{\alpha} = \frac{\beta}{x} \implies x = \alpha\beta.$$

Hence $\alpha\beta$ is constructible. With similar setting above, assuming $\left|\overline{AB}\right| = \beta$, $\left|\overline{AF}\right| = \alpha$(further assumption that $\alpha > 1$) and $\left|\overline{AD}\right| = x$, we have $x = \dfrac{\alpha}{\beta}$. Thus, $\dfrac{\alpha}{\beta}$ is also constructible. □

2. Prove that a line in F has an equation of the form $ax + by + c = 0$ with a, b, c in F.

Proof. Let $A(p, q)$ and $B(r, s)$ be the points in the plane of F. Let l denote the straight line passing through A and B. Then by the formula of line joining two points in the Euclidean plane, assuming $r \neq p$,

$$l : \frac{y - q}{s - q} = \frac{x - p}{r - p} \iff y = \frac{s - q}{r - p}x + \left(q - p \cdot \frac{s - q}{r - p}\right).$$

If $r = p$, it is just the line $x = p$. Hence in either cases, the line is of the form $ax+by+c = 0$, where $a, b, c \in F$. □

3. Prove that a circle in F has an equation of the form

$$x^2 + y^2 + ax + by + c = 0,$$

with $a, b, c \in F$.

Proof. Let $O(a, b)$ and $r \in F$ denote the centre and radius of a circle in the plane of F respectively. Then, it has the equation of the form

$$(x - a)^2 + (y - b)^2 = r^2 \iff x^2 + y^2 - 2ax - 2by + (a^2 + b^2 - r^2) = 0$$

so that it is the form of $x^2 + y^2 + ax + by + c = 0$, with $a, b, c \in F$. □

4. Prove that two lines in F, which intersect in the real plane, intersect at a point in the plane of F.

Proof. Let $l_1 : ax + by + c = 0$, $l_2 : px + qy + r = 0$ denote two lines in F. Suppose these two lines intersects. If $aq = bp$, this implies that l_1 and l_2 have same slopes and therefore, $l_1 = l_2$. If $aq \neq bp$, then the following linear system

$$\begin{pmatrix} a & b \\ p & q \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -c \\ -d \end{pmatrix} \iff \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{aq - bp} \begin{pmatrix} q & -b \\ -p & a \end{pmatrix} \begin{pmatrix} -c \\ -d \end{pmatrix}$$

leaves out with the unique solution such that(i.e., there is only one intersection)

$$x_0 = \frac{bd - cq}{aq - bp}, \quad y_0 = \frac{cp - ad}{aq - bp}.$$

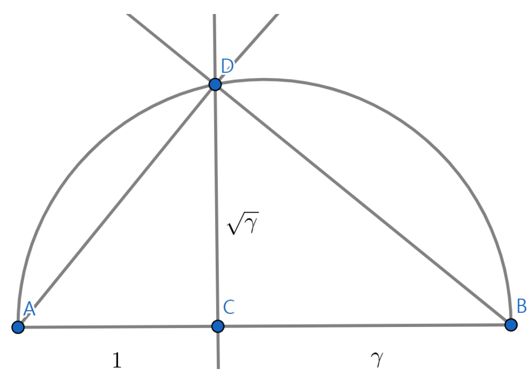
It is clear that both x_0 and y_0 are in F. Hence, l_1 and l_2 has an intersection in F provided that l_1 and l_2 intersects in the real plane. □

5. Prove that a line in F and a circle in F which intersect in the real plane do so at a point either in the plane of F or in the plane of $F(\sqrt{\gamma})$ where γ is a positive number in F.

Proof. Let $x^2 + y^2 + ax + by + c = 0$ and $px + qy + r = 0$ denote the equation of a circle and a line in the plane of F. Then equating the two equations gives a quadratic equation of x(or in y). Provided that the circle and line intersect in the real plane, the quadratic equation must yield solutions of the form $\alpha \pm \beta\sqrt{\gamma}$ for some $\alpha, \beta, \gamma(> 0) \in F$. Assuming one of the points of intersection is (x_0, y_0), we either have that $x_0, y_0 \in F$ if γ is a square, or $x_0, y_0 \in F(\sqrt{\gamma})$ otherwise. □

6. If $\gamma \in F$ is positive, prove that $\sqrt{\gamma}$ is realisable as an intersection of lines and circles in F.

Proof.



$Fig.2$

Observe Fig 2. We can construct a line segment $\overline{AB}$ with length $1+\gamma$ where $|\overline{AC}| = 1$ and $|\overline{CB}| = \gamma$. Further, draw a half circle having $|\overline{AB}|$ as its diameter. Draw a perpendicular line on C so that it intersects with D(Note that D is obtained as an intersection of a circle and a line in F). Now by the congruence of the triangle $\triangle ADC$ and $\triangle DBC$, we have that $|\overline{CD}| = \sqrt{\gamma}$. □

7. Prove that the following polynomials are irreducible over the field of rational numbers.
a) $8x^3 - 6x - 1$.

Proof. Substitute $x-1$ instead of x. Then we have

$$8(x-1)^3 - 6(x-1) - 1 = 8x^3 - 24x^2 + 18x - 3.$$

Now apply Eisenstein's criterion for $p = 3$. Thus we conclude that given polynomial is irreducible in $\mathbb{Q}$. □

b) $x^3 - 2$.

Proof. Apply Eisenstein's criterion for $p = 2$. □

c) $x^3 + x^2 - 2x - 1$.

Proof. Substitute $x+2$ instead of x. Then we have

$$(x+2)^3 + (x+2)^2 - 2(x+2) - 1 = x^3 + 7x^2 + 14x + 7.$$

Now apply Eisenstein's criterion for $p = 7$. Thus we conclude that given polynomial is irreducible in $\mathbb{Q}$. □

8. Prove that $2\cos(2\pi/7)$ satisfies $x^3 + x^2 - 2x - 1$.

Proof. Using that $2\cos(2\pi/7) = e^{\frac{2i\pi}{7}} + e^{-\frac{2i\pi}{7}}$,

$$\begin{aligned}(2\cos(2\pi/7))^3 + &(2\cos(2\pi/7))^2 - 2(2\cos(2\pi/7)) - 1\\ &= (e^{\frac{2i\pi}{7}} + e^{-\frac{2i\pi}{7}})^3 + (e^{\frac{2i\pi}{7}} + e^{-\frac{2i\pi}{7}})^2 - 2(e^{\frac{2i\pi}{7}} + e^{-\frac{2i\pi}{7}}) - 1\\ &= (e^{\frac{6i\pi}{7}} + 3e^{\frac{2i\pi}{7}} - 3e^{-\frac{2i\pi}{7}} + e^{-\frac{6i\pi}{7}}) + (e^{\frac{4i\pi}{7}} + 2 + e^{-\frac{4i\pi}{7}})\\ &\quad - 2(e^{\frac{2i\pi}{7}} + e^{-\frac{2i\pi}{7}}) - 1\\ &= 2\cos\frac{6\pi}{7} + 2\cos\frac{4\pi}{7} + 2\cos\frac{2\pi}{7} + 1\end{aligned}$$

where

$$\begin{aligned}2\cos\frac{6\pi}{7} + 2\cos\frac{4\pi}{7} + 2\cos\frac{2\pi}{7} + 1 &= \frac{\sin\frac{\pi}{7}}{\sin\frac{\pi}{7}}\left(2\cos\frac{6\pi}{7} + 2\cos\frac{4\pi}{7} + 2\cos\frac{2\pi}{7}\right) + 1\\ &= \frac{1}{\sin\frac{\pi}{7}}\left[\left(\sin\pi - \sin\frac{5\pi}{7}\right) + \left(\sin\frac{5\pi}{7} - \sin\frac{3\pi}{7}\right) + \left(\sin\frac{3\pi}{7} - \sin\frac{\pi}{7}\right)\right] + 1\\ &= -1 + 1 = 0.\end{aligned}$$

Therefore, $2\cos\frac{2\pi}{7}$ satisfies $x^3 + x^2 - 2x - 1$. □

9. Prove that the regular pentagon is constructible.

Proof. It is sufficient to show that $\zeta = e^{\frac{2\pi i}{5}}$ is a constructible number. We know that ζ is the standard 5th root of unity, so that $\zeta^5 = 1 \implies \zeta^4 + \zeta^3 + \zeta^2 + \zeta + 1 = 0$. Now from the fact that the polynomial $x^4 + x^3 + x^2 + x + 1$ is irreducible in $\mathbb{Q}$, $[\mathbb{Q}(\zeta) : \mathbb{Q}] = 4$. Now we see that ζ lies in the extension field of degree $4 = 2^2$, and hence, ζ is constructible. □

10. Prove that the regular hexagon is constructible.

Proof. It is sufficient to show that $\zeta = e^{\frac{\pi i}{3}}$ is a constructible number. It is clear that $\zeta = \frac{1}{2}$, and hence, $\zeta \in \mathbb{Q}$. It follows that ζ is constructible. □

11. Prove that the regular 15-gon is constructible.

Proof. We know that $e^{\frac{2\pi i}{5}}$ and $e^{\frac{\pi i}{3}}$ are constructible numbers. Moreover,

$$e^{\frac{2\pi i}{15}} = \left(\frac{e^{\frac{2\pi i}{5}}}{e^{\frac{\pi i}{3}}} \right)^2$$

so that $e^{\frac{2\pi i}{15}}$ is also constructible. Therefore, 15-gon is constructible. □

12. Prove that it is possible to trisect 72°.

Proof. It is sufficient to show that $e^{\frac{2\pi i}{5} \cdot \frac{1}{3}} = e^{\frac{2\pi i}{15}}$ is constructible. But we have proved this fact already in Problem 11. □

13. Prove that a regular 9-gon is not constructible.

Proof. Let $\zeta = e^{\frac{2\pi i}{9}}$. From $\zeta^9 = 1$, we know that

$$\zeta^6 + \zeta^3 + 1 = 0$$

and moreover, $f(x) = x^6 + x^3 + 1$ is irreducible in $\mathbb{Q}$. Thus, $[\mathbb{Q}(\zeta) : \mathbb{Q}] = 6$. But 6 is not a power of 2. Therefore, regular 9-gon is not constructible. □

14. Prove a regular 17-gon is constructible.

Proof. It is sufficient to show that $\zeta = e^{\frac{2\pi i}{17}}$ is a constructible number. It follows that $\zeta^{17} = 1 \implies \zeta^{16} + \zeta^{15} + \cdots + \zeta^2 + \zeta + 1 = 0$. From the fact that the polynomial $x^{16} + x^{15} + \cdots + x^2 + x + 1$ is irreducible in $\mathbb{Q}$, $[\mathbb{Q}(\zeta) : \mathbb{Q}] = 16$. Now we see that ζ lies in the extension field of degree $16 = 2^4$, and hence, ζ is constructible. □