Topics in Algebra solution

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Problems in Section 5.2.

1. Using the infinite series for e,

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{m!} + \cdots ,$$

prove that e is irrational.

Proof. Let s_n denote the partial sum $\sum_{k=0}^{n} \frac{1}{k!}$. We claim that $0 < e - s_n < \frac{1}{n!n}$. Observe that

$$\begin{aligned}
0 < e - s_n &= \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \cdots + \\
&= \frac{1}{n!}\left(\frac{1}{n+1} + \frac{1}{(n+2)(n+1)} + \cdots\right) \\
&< \frac{1}{n!}\left(\frac{1}{n+1} + \frac{1}{(n+1)^2} + \cdots\right) \\
&= \frac{1}{n!n}.
\end{aligned}$$

Suppose e was rational, then there exists positive integers $p, q \neq 0$ such that $e = p/q$. Then it follows that $0 < e - s_q < \frac{1}{q!q}$ so that $0 < q!(p/q - s_q) < 1/q$. But note that $q!(p/q - s_q) \in \mathbb{Z}$. Since there is no integer lies properly between 0 and $1/q$, it is a contradiction. Therefore, e is irrational. $\square$

2. If $g(x)$ is a polynomial with integer coefficients, prove that if p is a prime number then for $i \geq p$,

$$\frac{d^i}{dx^i}\left(\frac{g(x)}{(p-1)!}\right)$$

is a polynomial with integer coefficients each of which is divisible by p.

Proof. We prove by induction. Consider the base case $i = p$. The term x^i of $g(x)$, on differentiation by p times, vanishes if $i < p$ or leaves coefficient with multiple of $(i-1)!$ if $i \geq p$. Provided that $i \geq p$, $(i-1)!/(p-1)! = (i-p)(i-p-1)\cdots p$ and hence every coefficients of $\frac{d^p}{dx^p}g(x)/(p-1)!$ is differentiable by p. Now consider the statement is true for some $k > p$. That is,

$$\frac{d^k}{dx^k}\left(\frac{g(x)}{(p-1)!}\right) = \sum_{i=0}^{n} a_i x^i$$

where $p \mid a_i$. On differentiation,

$$\frac{d^{k+1}}{dx^{k+1}}\left(\frac{g(x)}{(p-1)!}\right) = \sum_{i=0}^{n} i a_i x^{i-1}.$$

Note that $p \mid ia_i$, since $p \mid a_i$. Thus, the $k+1$ th derivative of $g(x)/(p-1)!$ has coefficients divisible by p. Now by induction, this holds for all $k \geq p$. $\square$

3. If a is any real number, prove that $(a^m/m!) \to 0$ as $m \to \infty$.

Proof. Let $x_n = a^n/n!$. It is enough to show that $|x_{n+1}/x_n| \to 0$. Observe that

$$\left|\frac{x_{n+1}}{x_n}\right| = \frac{|a|}{n+1} \to 0 \quad \text{as } n \to \infty.$$

Therefore, $x_n \to 0$ as $n \to \infty$. $\square$

4. If $m > 0$ and n are integers, prove that $e^{m/n}$ is transcendental.

Proof. We first prove that $e^{1/n}$ is transcendental. For the sake of contradiction, was it to be algebraic, then $(e^{1/n})^n = e$ must be algebraic. But this contradicts the transcendental nature of e. Now we assume that $e^{m/n}$ is algebraic for $m > 0$. But we know that if a complex number α^m is algebraic, where $m > 0$ is a positive integer, then so does α. So this also leads us to conclude that e is algebraic, which is a contradiction again. Therefore, $e^{m/n}$ is transcendental for all integers $m > 0, n \in \mathbb{Z}$. $\square$