Topics in Algebra solution

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Problems in the Section 2.6.

1. If H is a subgroup of G such that the product of two right cosets of H in G is again a right coset of H in G, prove that H is normal in G.

Proof. Consider the product of Hg and Hg^{-1}. Consequently, $HgHg^{-1} = Hc$ for some $c \in G$. As $egeg^{-1} = e = hc$ for some $h \in H$, $c \in H$. Therefore, $HgHg^{-1} = H$, implying $gHg^{-1} = H$. Hence H is normal in G. □

2. If G is a group and H is a subgroup of index 2 in G, prove that H is a normal subgroup of G.

Proof. We can represent the coset decomposition of G in two different ways. That is,

$$G = H \amalg xH, \quad G = H \amalg Hx$$

for some $x \notin H$. This forces us that $xH = Hx$, hence H is normal in G. □

3. If N is a normal subgroup of G and H is any subgroup of G, prove that NH is a subgroup of G.

Proof. Let $nh, n'h' \in NH$. Consequently,

$$nh \cdot n'h' = nhn'h^{-1}hh' = n(n'')hh' \in NH$$

and

$$(nh)^{-1} = h^{-1}n^{-1} = h^{-1}n^{-1}hh^{-1} = n'h^{-1} \in NH.$$

Hence, NH is a subgroup of G. □

4. Show that the intersection of two normal subgroups of G is a normal subgroup of G.

Proof. Let N and M be the two normal subgroups of G. We know that

$$g(N \cap M)g^{-1} = gNg^{-1} \cap gMg^{-1} = N \cap M.$$

Hence, $N \cap M$ is normal in G. □

5. If H is a subgroup of G and N is a normal subgroup of G, show that $H \cap N$ is a normal subgroup of H.

Proof. Let $g \in H \cap N$. Then for any $h \in H$, $hgh^{-1} \in H$ since $g \in H$, and $hgh^{-1} \in H$ since $g \in N$ and N is normal in G. Thus, $hgh^{-1} \in H \cap N$, and $H \cap N$ is normal in H. □

6. Show that every subgroup of abelian group is normal.

Proof. Let H be a subgroup of an abelian group G. Then for any $g \in G$, $h \in H$, $ghg^{-1} = gg^{-1}h = h \in H$ implying H is normal. □

7. Is the converse of Problem 6 true? If yes, prove it, if no, give an example of a non-abelian group all of whose subgroups are normal.

Proof. Converse of Problem 6 is False. Consider the group of quaternions, $G = \{\pm 1, \pm i, \pm j, \pm k\}$. There are 4 non-trivial subgroups : $\{\pm 1, \pm i\}$, $\{\pm 1, \pm j\}$, $\{\pm 1, \pm k\}$ and $\{\pm 1\}$. These are all normal in G, but G is not abelian since i and j does not commute. □

8. Give an example of a group G, subgroup H, and an element $a \in G$ such that $aHa^{-1} \subset H$ but $aHa^{-1} \neq H$.

Proof. Let G be the multiplicative group of 2×2 real matrices. Consider the subgroup $H = \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right) = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \right\}$ of G. Take $g = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \in G$. Then

$$gHg^{-1} = \left\{ \begin{pmatrix} 1 & 2n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \right\} \subset H$$

but $gHg^{-1} \neq H$. □

9. Suppose H is the only subgroup of order $o(H)$ in the finite group G. Prove that H is a normal subgroup of G.

Proof. Note that for all $g \in G$, gHg^{-1} is a subgroup of G and $o(gHg^{-1}) = o(H)$. Therefore, $gHg^{-1} = H$ for all $g \in G$ and hence H is normal in G. □

10. If H is a subgroup of G, let $N(H) = \{g \in G : gHg^{-1} = H\}$. Prove
a) $N(H)$ is a subgroup of G.

Proof. Let $a, b \in N(H)$. Consequently, $(ab)H(ab)^{-1} = a(bHb^{-1})a^{-1} = aHa^{-1} = H$ so that $ab \in N(H)$. Also, $aHa^{-1} = H \implies H = a^{-1}Ha$ so that $a^{-1} \in H$. Hence $N(H)$ is a subgroup of G. □

b) H is normal in $N(H)$.

Proof. It is trivial by the definition of $N(H)$. □

c) If H is a normal subgroup of K in G, then $K \subset N(H)$ (that is, $N(H)$ is the largest subgroup of G in which H is normal.

Proof. Let $k \in K$. Then $kHk^{-1} = H$ implying $k \in N(H)$. Hence $K \subset N(H)$. □

d) H is normal in G if and only if $N(H) = G$.

Proof. It is clear that $N(H) \subset G$. So, we show that the other side of inclusion holds if H is normal in G. But this is also trivial to check since H is normal in G, $gHg^{-1} = H$ for all $g \in G$ and hence $g \subset N(H)$. Thus, $N(H) = G$. Moreover, if $N(H) = G$, this itself implies that $gHg^{-1} = H$ for all $g \in G$ and so that H is normal in G. □

11. If N and M are normal subgroups of G, prove that NM is also a normal subgroup of G.

Proof. Note that for all $g \in G$,

$$g(NM)g^{-1} = (gNg^{-1})(gMg^{-1}) = NM$$

so that NM is a normal subgroup of G. □

12. Suppose that N and M are two normal subgroups of G and that $N \cap M = (e)$. Show that for any $n \in N, m \in M$, $nm = mn$.

Proof. Let $n \in N, m \in M$. Note that $nm = (mm^{-1})nm = m(m^{-1}nm) = mn'$ for some $n' \in N$. We shall show that n' is, in fact, equals n exactly. Observe that

$$\begin{aligned} nm(n')^{-1} = m &\implies nmn^{-1}n(n')^{-1} = m \\ &\implies m'n(n')^{-1} = m \quad \text{(for some } m' \in M) \\ &\implies n(n')^{-1} = (m')^{-1}m \in M \\ &\implies n = n' \quad (\because N \cap M = (e)). \end{aligned}$$

Hence, $nm = mn$ for all $n \in N, m \in M$. □

13. If a cyclic subgroup T of G is normal in G, then show that every subgroup of T is normal in G.

Proof. Since T is cyclic, $T = (a)$ for some $a \in G$. Let H be a subgroup of T. Since H is also cyclic, $H = (a^k)$ for some integer $k \leq 0$. Choose an element $(a^k)^h = a^{kh} \in H$. Then for all $g \in G$, $gag^{-1} \in T$. Let $gag^{-1} = a^l$. Further, $ga^{kh}g^{-1} = (gag^{-1})^{kh} = (a^l)^{kh} = (a^k)^{lh} \in H$. This shows that H is normal in G. □

14. Prove, by an example, that we can find three groups $E \subset F \subset G$, where E is normal in F, F is normal in G, but E is not normal in G.

Proof. Let $G = S_4$, $F = \{id, (12)(34), (13)(24), (14)(23)\}$, $E = ((12)(34))$. F is normal in G as for any conjugate of elements of the form $(a, b)(c, d)$ results out with the same. And also $[F : E] = 2$, E is normal in F. But E is clearly not normal in G. □

15. If N is normal in G and $a \in G$ is of order $o(a)$, prove that the order, m, of Na in G/N is a divisor of $o(a)$.

Proof. Let $k = o(a)$. If $m \nmid k$, then $k = mq + r$ for some integer q, r such that $0 \leq r < m$. Note that $a^m \in N$ and $a^k = e$. Consequently,

$$a^k = a^{mq+r} = (a^m)^q \cdot a^r \implies a^r \in N.$$

But since m is the order of Na in G/N and $r < m$, this is a contradiction. □

16. If N is a normal subgroup in the finite group such that $i_G(N)$ and $o(N)$ are relatively prime, show that any element $x \in G$ satisfying $x^{o(N)} = e$ must be in N.

Proof. Since $\gcd(i_G(N), o(N) = 1$, $i_G(N)\lambda + o(N)\mu = 1$ for some integers $\lambda, \mu \in \mathbb{Z}$. Note that $x^{i_G(N)} \in N$. Consequently,

$$x = x^{i_G(N)\lambda + o(N)\mu} = x^{i_G(N)\lambda} \in N.$$

□

17. Let G be defined as all formal symbols $x^i y^j, i = 0, 1, j = 0, 1, 2, \cdots, n-1$ where we assume

$$\begin{aligned} &x^i y^j = x^{i'} y^{j'} \quad \text{if and only if} \quad i = i', j = j' \\ &x^2 = y^n = e, \quad n > 2 \\ &xy = y^{-1}x \end{aligned}$$

a) Find the form of the product $(x^i y^j)(x^k y^l)$ as $x^\alpha y^\beta$.

Proof. Using the fact that $xy^m = y^{-m}x$, we have the following:

$$(x^iy^j)(x^ky^l) = \begin{cases} y^{j+l \pmod n}, & i,k \equiv 0 \pmod 2 \\ xy^{j+l \pmod n}, & i \equiv 1, k \equiv 0 \pmod 2 \\ xy^{l-j \pmod n}, & i \equiv 0, k \equiv 1 \pmod 2 \\ y^{l-j \pmod n}, & i,k \equiv 1 \pmod 2). \end{cases}$$

In an ONE-line presentation, we have:

$$(x^iy^j)(x^ky^l) = x^{i+k \pmod 2} y^{((-1)^{k \pmod 2} j + l) \pmod n}.$$

□

b) Using this, prove that G is a non-abelian group of order $2n$.

Proof. In general, if $n > 2$, $y^{-1} \neq y$. Since $xy = y^{-1}x$, $yx \neq y^{-1}x$ implying $xy \neq yx$. Hence G is a non-abelian group of order $2n$. □

c) If n is odd, prove that the center of G is (e), while if n is even, the center of G is larger than (e).

Proof. We make a case-by-case investigation. First, for the elements of the form xy^m, $m > 1$, note that $x(xy^m)x^{-1} = xxy^mx = y^mx = xy^{-m}$. But since n is odd, Suppose that $xy^m = xy^{-m}$. It is must then that $2m \equiv 0 \pmod n$, implying n is even thereby yielding a contradiction. Now consider the elements of the form y^m. Also, $xy^mx = y^{-m}$. Suppose that $y^m = y^{-m}$, similarly above, n must be even, contradiction. Finally, we consider x. Then $y^{-1}xy = xy^2$. But for $n > 2$, $y^2 \neq e$. Hence, $xy^m, y^m, x (m > 1)$ are not in the center of G if $n > 2$ is odd. Thus, $Z(G) = (e)$.
Now suppose $n > 2$ is even. Then it is easy to see that $y^{n/2} \in Z(G)$. Hence, $Z(G) \supsetneq (e)$. □

18. Let G be a group in which, for some integer $n > 1$, $(ab)^n = a^nb^n$ for all $a, b \in G$. Show that
a) $G^{(n)} = \{x^n : x \in G\}$ is a normal subgroup of G.

Proof. We first show that $G^{(n)}$ is a subgroup of G. Let $a^n, b^n \in G^{(n)}$. Then $a^nb^n = (ab)^n \in G^{(n)}$. Also, $(a^n)^{-1} = (a^{-1})^n \in G^{(n)}$. Hence, $G^{(n)}$ is a subgroup of G. It is also normal in G, as, for all $g \in G$, $a^n \in G^{(n)}$, $ga^ng^{-1} = (gag^{-1})^n \in G^{(n)}$. □

b) $G^{(n-1)} = \{x^{n-1} : x \in G\}$ is a normal subgroup of G.

Proof. Note that $a^{n-1}b^{n-1} = (ba)^{n-1}$. From this, we see that $G^{(n-1)}$ is a subgroup of G. Normality is clear, since for any $g \in G$, $ga^{n-1}g^{-1} = (gag^{-1})^{n-1} \in G^{(n-1)}$. □

19. Let G be as in Problem 18. Show
a) $a^{n-1}b^n = b^n a^{n-1}$ for all $a, b \in G$.

Proof. From

$$(ba)^{n-1} = a^{n-1}b^{n-1}, \quad (ab)^{n-1} = b^{n-1}a^{n-1},$$

we have

$$a^{n-1}b^n = a^{n-1}b^{n-1}b = (ba)^{n-1}b = b(ab)^{n-1} = bb^{n-1}a^{n-1} = b^n a^{n-1}$$

for all $a, b \in G$. □

b) $(aba^{-1}b^{-1})^{n(n-1)} = e$ for all $a, b \in G$

Proof. Note that

$$\begin{aligned} a^{n(n-1)}b^{n(n-1)} &= (a^{n-1}b^{n-1})^n = (a^{n-1}b^n b^{-1})^n \\ &= (b^n a^{n-1} b^{-1})^n = b^{n^2}(a^n)^{n-1}(b^{-1})^n \\ &= b^{n^2}(b^{-1})^n(a^n)^{n-1} = b^{n(n-1)}a^{n(n-1)} \end{aligned}$$

implying

$$a^{n(n-1)}b^{n(n-1)} \cdot (a^{-1})^{n(n-1)}(b^{-1})^{n(n-1)} = (aba^{-1}b^{-1})^{n(n-1)} = e.$$

□

20. Let G be a group such that $(ab)^p = a^p b^p$ for all $a, b \in G$, where p is a prime number. Let $S = \{x \in G : x^{p^m} = e \text{ for some } m \text{ depending on } x\}$. Prove
a) S is a normal subgroup of G.

Proof. We first prove that S is a subgroup of G. First, we note that $(ab)^{p^n} = a^{p^n}b^{p^n}$. This can be shown easily by induction process. Suppose $a, b \in S$, where $a^{p^n} = e, b^{p^m} = e$ for some $n, m \in \mathbb{Z}$. Then

$$(ab)^{p^{mn}} = a^{p^{mn}}b^{p^{mn}} = (a^{p^n})^m(b^{p^m})^n = e$$

so that $ab \in S$. Also, $(a^{-1})^{p^n} = (a^{p^n})^{-1} = e$. Hence, S is a subgroup of G. Normality is also clear, since for all $g \in G$, $(gag^{-1})^{p^n} = ga^{p^n}g^{-1} = e$ implying $gag^{-1} \in S$. □

b) If $\overline{G} = G/S$ and if $\overline{x} \in \overline{G}$ is such that $\overline{x}^p = \overline{e}$ then $\overline{x} = \overline{e}$.

Proof. Note that $\overline{x}^p = \overline{e}$ implies $x^p \in S$. Equivalently, $(x^p)^{p^k} = e$ for some $k \in \mathbb{Z}$. In fact, $(x^p)^{p^k} = x^{p^{k+1}} = e \implies x \in S$, so that $\overline{x} = \overline{e}$. □

21. Let G be the set of all real 2×2 matrices $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ where $ad \neq 0$ under matrix multiplication. Let $N = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \right\}$. Prove that

a) N is a normal subgroup of G

Proof. Let $g = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in G$, $n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \in N$. Then

$$gng^{-1} = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{a} & -\frac{b}{ad} \\ 0 & \frac{1}{d} \end{pmatrix} = \begin{pmatrix} 1 & \frac{an}{d} \\ 0 & 1 \end{pmatrix} \in N$$

so that N is normal in G. □

b) G/N is abelian.

Proof. Let $g_1 = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in G$, $g_2 = \begin{pmatrix} p & q \\ 0 & r \end{pmatrix}$ $n = \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \in N$. Set $s = c + \dfrac{b(p-r) + q(d-a)}{ap}$. Then we see that

$$g_1 g_2 n \in g_1 g_2 N, g_1 g_2 n = g_2 g_1 m \in g_2 g_1 N$$

where $m = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \in N$. Similarly, we can show the opposite inclusion. Thus, G/N is an abelian group. □