Topics in Algebra solution

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Problems in Section 2.14.

1. If G is an abelian group of order p^n, p a prime and $n_1 \geq n_2 \geq \cdots \geq n_k > 0$, are the invariants of G, show that the maximal order of any elements in G is p^{n_1}.

Proof. Choose $g \in G$. Consequently, $g = x_1^{a_1} x_2^{a_2} \cdots x_k^{a_k}$, where $a_i \leq p^{n_i}$ for each i. It is enough to show that $g^{p^{n_1}} = 0$. Observe that

$$\begin{aligned} g^{p^{n_1}} &= (x_1^{a_1} x_2^{a_2} \cdots x_k^{a_k})^{p^{n_1}} \\ &= x_1^{a_1 \cdot p^{n_1}} x_2^{a_2 \cdot p^{n_1}} \cdots x_k^{a_k \cdot p^{n_1}} \\ &= (x_1^{p_{n_1}})^{a_1} (x_2^{p_{n_2}})^{p_{n_1 - n_2} a_2} \cdots (x_k^{p_{n_k}})^{p_{n_1 - n_k} a_k} \\ &= e \cdot e \cdots e = e \end{aligned}$$

so that every element of G has maximal order of p^{n_1}. □

2. If G is a group, $A_1, \cdots, A_k$ normal subgroups of G such that $A_i \cap (A_1 A_2 \cdots A_{i-1}) = (e)$ for all i, show that G is the direct product of $A_1, \cdots A_k$ if $G = A_1 \cdots A_k$.

Proof. From that

$$o(A_1 A_2 \cdots A_{k-1} A_k) = \frac{\prod_{i=1}^{k} o(A_i)}{\prod_{n=1}^{k-1} o((\prod_{j=1}^{k-n} A_j) \cap A_{k+1-n})},$$

since $A_i \cap (A_1 A_2 \cdots A_{i-1}) = (e)$ for all i,

$$o(G) = o(A_1 A_2 \cdots A_{k-1} A_k) = \prod_{i=1}^{k} o(A_i).$$

Now by applying the result of Problem 18 of Section 2.13, G is the direct product of $A_1, \cdots A_k$. □

3. Using Theorem 2.14.1, prove that if a finite abelian group has subgroups of order m and n, then it has a subgroup whose order is the least common multiple of m and n.

Proof. We know that if a finite abelian group G exists and d is a positive integer such that d divides $o(G)$, then there is a subgroup of G of order d. Now, we set H and K be the subgroup of G with orders m and n respectively. Then we have

$$o(HK) = \frac{o(H)o(K)}{o(H \cap K)}.$$

Note that by Lagrange's theorem, $o(H \cap K)$ divides both $o(H) = m, o(K) = n$. Hence, $\gcd(m,n)|o(H \cap K)$. Moreover, $\text{lcm}(m,n)|o(HK)$ from the above identity. Since HK is abelian, there exists a subgroup of HK(hence of G) of order $\text{lcm}(m,n)$. □

4. Describe all the finite abelian groups of order
a) 2^6

Solution. A finite abelian group of order 2^6 is isomorphic to one of the 11 below:

$$\begin{aligned}
&\mathbb{Z}_{2^6}, \quad \mathbb{Z}_{2^5} \times \mathbb{Z}_2, \quad \mathbb{Z}_{2^4} \times \mathbb{Z}_{2^2}, \quad \mathbb{Z}_{2^4} \times \mathbb{Z}_2 \times \mathbb{Z}_2,\\
&\mathbb{Z}_{2^3} \times \mathbb{Z}_{2^3}, \quad \mathbb{Z}_{2^3} \times \mathbb{Z}_{2^2} \times \mathbb{Z}_2, \quad \mathbb{Z}_{2^3} \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2,\\
&\mathbb{Z}_{2^2} \times \mathbb{Z}_{2^2} \times \mathbb{Z}_{2^2}, \quad \mathbb{Z}_{2^2} \times \mathbb{Z}_{2^2} \times \mathbb{Z}_2 \times \mathbb{Z}_2 \quad \mathbb{Z}_{2^2} \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2,\\
&\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2.
\end{aligned}$$

□

b) 11^6

Solution. A finite abelian group of order 11^6 is isomorphic to one of the 11 below:

$$\begin{aligned}
&\mathbb{Z}_{11^6}, \quad \mathbb{Z}_{11^5} \times \mathbb{Z}_{11}, \quad \mathbb{Z}_{11^4} \times \mathbb{Z}_{11^2}, \quad \mathbb{Z}_{11^4} \times \mathbb{Z}_{11} \times \mathbb{Z}_{11},\\
&\mathbb{Z}_{11^3} \times \mathbb{Z}_{11^3}, \quad \mathbb{Z}_{11^3} \times \mathbb{Z}_{11^2} \times \mathbb{Z}_{11}, \quad \mathbb{Z}_{11^3} \times \mathbb{Z}_{11} \times \mathbb{Z}_{11} \times \mathbb{Z}_{11},\\
&\mathbb{Z}_{11^2} \times \mathbb{Z}_{11^2} \times \mathbb{Z}_{11^2}, \quad \mathbb{Z}_{11^2} \times \mathbb{Z}_{11^2} \times \mathbb{Z}_{11} \times \mathbb{Z}_{11} \quad \mathbb{Z}_{11^2} \times \mathbb{Z}_{11} \times \mathbb{Z}_{11} \times \mathbb{Z}_{11},\\
&\mathbb{Z}_{11} \times \mathbb{Z}_{11} \times \mathbb{Z}_{11} \times \mathbb{Z}_{11} \times \mathbb{Z}_{11} \times \mathbb{Z}_{11}.
\end{aligned}$$

□

c) 7^5

Solution. A finite abelian group of order 7^5 is isomorphic to one of the 7 below:

$$\begin{aligned}
&\mathbb{Z}_{7^5}, \quad \mathbb{Z}_{7^4} \times \mathbb{Z}_7, \quad \mathbb{Z}_{7^3} \times \mathbb{Z}_{7^2}, \quad \mathbb{Z}_{7^3} \times \mathbb{Z}_7 \times \mathbb{Z}_7,\\
&\mathbb{Z}_{7^2} \times \mathbb{Z}_{7^2} \times \mathbb{Z}_7, \quad \mathbb{Z}_{7^2} \times \mathbb{Z}_7 \times \mathbb{Z}_7 \times \mathbb{Z}_7, \quad \mathbb{Z}_7 \times \mathbb{Z}_7 \times \mathbb{Z}_7 \times \mathbb{Z}_7 \times \mathbb{Z}_7.
\end{aligned}$$

□

d) $2^4 \cdot 3^4$

Solution. A finite abelian group of order $2^4 \cdot 3^4$ is isomorphic to one of the $5 \cdot 5 = 25$ below:

$\mathbb{Z}_{2^4} \times \mathbb{Z}_{3^4}$, $\mathbb{Z}_{2^3} \times \mathbb{Z}_2 \times \mathbb{Z}_{3^4}$, $\mathbb{Z}_{2^2} \times \mathbb{Z}_{2^2} \times \mathbb{Z}_{3^4}$, $\mathbb{Z}_{2^2} \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{3^4}$,
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{3^4}$, $\mathbb{Z}_{2^4} \times \mathbb{Z}_{3^3} \times \mathbb{Z}_3$, $\mathbb{Z}_{2^3} \times \mathbb{Z}_2 \times \mathbb{Z}_{3^3} \times \mathbb{Z}_3$, $\mathbb{Z}_{2^2} \times \mathbb{Z}_{2^2} \times \mathbb{Z}_{3^3} \times \mathbb{Z}_3$,
$\mathbb{Z}_{2^2} \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{3^3} \times \mathbb{Z}_3$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{3^3} \times \mathbb{Z}_3$, $\mathbb{Z}_{2^4} \times \mathbb{Z}_{3^2} \times \mathbb{Z}_{3^2}$,
$\mathbb{Z}_{2^3} \times \mathbb{Z}_2 \times \mathbb{Z}_{3^2} \times \mathbb{Z}_{3^2}$, $\mathbb{Z}_{2^2} \times \mathbb{Z}_{2^2} \times \mathbb{Z}_{3^2} \times \mathbb{Z}_{3^2}$, $\mathbb{Z}_{2^2} \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{3^2} \times \mathbb{Z}_{3^2}$,
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{3^2} \times \mathbb{Z}_{3^2}$, $\mathbb{Z}_{2^4} \times \mathbb{Z}_{3^2} \times \mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathbb{Z}_{2^3} \times \mathbb{Z}_2 \times \mathbb{Z}_{3^2} \times \mathbb{Z}_3 \times \mathbb{Z}_3$,
$\mathbb{Z}_{2^2} \times \mathbb{Z}_{2^2} \times \mathbb{Z}_{3^2} \times \mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathbb{Z}_{2^2} \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{3^2} \times \mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{3^2} \times \mathbb{Z}_3 \times \mathbb{Z}_3$,
$\mathbb{Z}_{2^4} \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathbb{Z}_{2^3} \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathbb{Z}_{2^2} \times \mathbb{Z}_{2^2} \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$,
$\mathbb{Z}_{2^2} \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$.

□

5. Show how to get all abelian groups of order $2^3 \cdot 3^4 \cdot 5$.

Solution. We make use of the Corollary of Theorem 2.14.3. That is, the number of nonisomorphic abelian groups of order $p_1^{a_1} \cdots p_r^{a_r}$, where p_i are distinct primes with $a_i > 0$, is $p(a_1) \cdots p(a_r)$, where $p(u)$ denotes the number of partitions of u. In our cases, we have total of $p(3) \cdot p(4) \cdot p(1) = 15$ nonisomporhic abelian copies. Explicitly,

$\mathbb{Z}_{2^3} \times \mathbb{Z}_{3^4} \times \mathbb{Z}_5$, $\mathbb{Z}_{2^2} \times \mathbb{Z}_2 \times \mathbb{Z}_{3^4} \times \mathbb{Z}_5$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{3^4} \times \mathbb{Z}_5$,
$\mathbb{Z}_{2^3} \times \mathbb{Z}_{3^3} \times \mathbb{Z}_3 \times \mathbb{Z}_5$, $\mathbb{Z}_{2^2} \times \mathbb{Z}_2 \times \mathbb{Z}_{3^3} \times \mathbb{Z}_3 \times \mathbb{Z}_5$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{3^3} \times \mathbb{Z}_3 \times \mathbb{Z}_5$,
$\mathbb{Z}_{2^3} \times \mathbb{Z}_{3^2} \times \mathbb{Z}_{3^2} \times \mathbb{Z}_5$, $\mathbb{Z}_{2^2} \times \mathbb{Z}_2 \times \mathbb{Z}_{3^2} \times \mathbb{Z}_{3^2} \times \mathbb{Z}_5$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{3^2} \times \mathbb{Z}_{3^2} \times \mathbb{Z}_5$,
$\mathbb{Z}_{2^3} \times \mathbb{Z}_{3^2} \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5$, $\mathbb{Z}_{2^2} \times \mathbb{Z}_2 \times \mathbb{Z}_{3^2} \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5$,
$\mathbb{Z}_{2^3} \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{3^2} \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5$,
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5$, $\mathbb{Z}_{2^2} \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5$.

□

6. If G is an abelian group of order p^n with invariants $n_1 \geq n_2 \geq n_k > 0$ and $H \neq (e)$ is a subgroup of G, show that if $h_1 \geq h_2 \geq h_s > 0$ are the invariants of H, then $k \geq s$ and for each i, $h_i \leq n_i$ for $i = 1, 2, \cdots, s$.

Proof. We know that G is an internal product expressed as $G = A_1 A_2 \cdots A_k$, where A_i are normal subgroups of order p^{n_i} each. Consequently,

$$H = H \cap G = H \cap (A_1 A_2 \cdots A_k) = \prod_{i=1}^{k} (H \cap A_i) = B_1 B_2 \cdots B_s$$

where $H \cap A_i = B_i$ for each $i = 1, \cdots, s, H \cap A_i \neq (e)$. Clearly, H is internal products of B_i's and each B_i has order p^{h_i}, with $h_i \leq n_i$. $k \geq s$ is trivial now. □

If G is an abelian group, let $\widehat{G}$ be the set of all homomorphisms of G into the group of non-zero complex numbers under multiplication. If $\phi_1, \phi_2 \in \widehat{G}$, define $\phi_1 \cdot \phi_2$ by $(\phi_1 \cdot \phi_2)(g) = \phi_1(g)\phi_2(g)$ for all $g \in G$.

7. Show that $\widehat{G}$ is an abelian group under the operation defined.

Proof. Choose any $\phi_1, \phi_2 \in \widehat{G}$. Then for all $g \in G$, as multiplication in complex numbers is commutative, $\phi_1(g)\phi_2(g) = \phi_2(g)\phi_1(g)$. Hence, $\widehat{G}$ is abelian. $\square$

8. If $\phi \in \widehat{G}$ and G is finite, show that $\phi(g)$ is a root of unity for every $g \in G$.

Proof. If G is finite, then for every $g \in G$, there is an integer $k = o(G)$ so that $g^k = e$. Consequently, $\phi(g)^k = \phi(g^k) = 1$ so that $\phi(g)$ is a root of unity in $\mathbb{C}$. $\square$

9. If G is finite cyclic group, show that $\widehat{G}$ is cyclic and $o(\widehat{G}) = o(G)$, hence G and $\widehat{G}$ are isomorphic.

Proof. Let $o(G) = n$. Suppose $G = (g)$. Then for all $\phi \in \widehat{G}$, $\phi(g)^n = 1$ so that $\phi(G)$ is mapped into the subgroup of roots of unity of n. Choose $\phi \in \widehat{G}$ such that $\phi(g) = w$. It clearly has order n. For any $\psi \in \widehat{G}$, $\psi(g^a)$ is mapped to (w) in $\mathbb{C}$, so that $\psi \in (\phi)$. Thus, $\widehat{G}$ is cyclic, with order n and hence G and $\widehat{G}$ are isomorphic. $\square$

10. if $g_1 \neq g_2$ are in G, G a finite abelian group, prove that there is a $\phi \in \widehat{G}$ with $\phi(g_1) \neq \phi(g_2)$.

Proof. Since every finite abelian group is a product of cyclic groups, it is sufficient to consider only the cyclic case. But for any finite cyclic group, by Problem 9, $\widehat{G} \simeq G$ so that for all $g \neq e \in G$, there is $\phi \in \widehat{G}$ such that $\phi(g) \neq 1$. Hence, $\phi(g_1 g_2) \neq 1 \iff \phi(g_1) \neq \phi(g_2)$. $\square$

11. If G is a finite abelian group prove that $o(G) = o(\widehat{G})$ and G is isomorphic to $\widehat{G}$.

Proof. Note that every finite abelian group is the direct product of cyclic groups. Since for every finite cyclic group H, $H \simeq \widehat{H}$, so that $G \simeq \widehat{G}$. $\square$

12. If $\phi \neq 1 \in \widehat{G}$ where G is an abelian group, show that $\sum_{g \in G} \phi(g) = 0$.

Proof. Since G is abelian, we can take $b \in G$ such that $\phi(b) \neq 1$ for all $\phi \in \widehat{G}$. Thus,

$$\sum_{g \in G} \phi(g) = \sum_{g \in G} \phi(gb) = \phi(b) \sum_{g \in G} \phi(g)$$

which implies $(1 - \phi(b)) \sum_{g \in G} \phi(g) = 0$ so that $\sum_{g \in G} \phi(g) = 0$. $\square$