Topics in Algebra solution

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Problems in Section 2.12.

1. Adapt the second proof given of Sylow's theorem to prove directly that if p is a prime and $p^\alpha \mid o(G)$, then G has a subgroup of order p^α.

Proof. If the order of group is 2, then the result is trivial. So we assume that the result to be true for all groups of order less than $o(G)$. Suppose there is a subgroup $H \neq G$ such that $p^\alpha \mid o(H)$, then we are done. So we assume that there is no $H \neq G$, a subgroup of G satisfying $p^\alpha \nmid o(H)$. Recall the class equation which states that

$$o(G) = z + \sum_{a \notin Z} \frac{o(G)}{o(N(a))}$$

where $z = o(Z)$, Z the centre of G. Since $p^\alpha \mid o(G)$ but $p^\alpha \nmid o(H)$,

$$p \mid \frac{o(G)}{o(N(a))} \implies p \mid \sum_{a \notin Z} \frac{o(G)}{o(N(a))} \implies p \mid z.$$

Now by applying Cauchy's theorem on $Z(G)$, there is an $b \in Z$ with order p. Let $B = (b)$. Consequently, G/B is a group of order with $p^{\alpha-1}$ as a divisor. By our induction hypothesis, there is a subgroup $\overline{P}$ of G/B with order $p^{\alpha-1}$. Let $P = \{x \in G : xZ \in \overline{P}\}$. Then $\overline{P} \simeq P/B$, so that P is a subgroup of G with order $p^{(\alpha-1)+1} = p^\alpha$. Now our induction process is completed and P is the desired subgroup of order p^α. $\square$

2. If $x > 0$ is a real number, define $[x]$ to be m, where m is that integer such that $m \leq x < m+1$. If p is a prime, show that the power of p which exactly divides $n!$ is given by

$$\left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \cdots + \left[\frac{n}{p^r}\right] + \cdots.$$

Proof. The power of p which exactly divides $n!$ equals to the power of p in prime factorization of $n!$. That is, the number of times that p divdes $n!$. Note that $\left[\frac{n}{p^r}\right]$ is the number of distinct integers less than n, which are multiples of p^r. Thus, the sum $\left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \cdots + \left[\frac{n}{p^r}\right] + \cdots$ gives the total number of times that p divides $n!$. Hence proved. □

3. Use the method for constructing the p-Sylow subgroup of S_{p^k} to find generators for a) a 2-Sylow subgroup in S_8.

Solution. We follow the method used in Lemma 2.12.2. By dividing $\{1, 2, \cdots, 8\}$ into 2 clumps:

$$\{1, 2, 3, 4\}, \{5, 6, 7, 8\},$$

let $\sigma = (1, 5)(2, 6)(3, 7)(4, 8)$. Set P_1 be a subgroup of order $2^{n(2)} = 2^3 = 8$ isomorphic to a subgroup of S_4, that is, $P_1 = ((1, 2, 3, 4), (1, 3))$. Let $P_2 = \sigma^{-1}P_1\sigma = ((5, 6, 7, 8), (5, 7))$. Set $T = P_1P_2$. Note that $\sigma^2 = e, \sigma \notin T$ and $\sigma^{-1}T\sigma = T$. Let $P = \{\sigma^j \cdot t : t \in T, 0 \leq j \leq 1\}$. Then P is a subgroup of order $2 \cdot 2^6 = 2^7$, an 2-Sylow subgroup of S_8. i.e,

$$P = (\{(1, 3), (1, 2, 3, 4), (1, 5)(2, 6)(3, 7)(4, 8)\})$$

is an 2-Sylow subgroup of S_8. □

b) a 3-Sylow subgroup in S_9.

Solution. By dividing $\{1, 2, 3, \cdots, 8, 9\}$ into 3 clumps:

$$\{1, 2, 3\},\ \{4, 5, 6\},\ \{7, 8, 9\},$$

let $\sigma = (1, 4, 7)(2, 5, 8)(3, 6, 9)$. Set P_1 be a subgroup of order $3^{n(1)} = 3$ isomorphic to a subgroup of S_3, that is, $P_1 = ((1, 2, 3))$. Let $P_2 = \sigma^{-1}P_1\sigma = ((4, 5, 6))$, $P_3 = \sigma^{-2}P_1\sigma^2 = ((7, 8, 9))$. Set $T = P_1P_2P_3$. Note that $\sigma^3 = e, \sigma \notin T$ and $\sigma^{-1}T\sigma = T$. Let $P = \{\sigma^j \cdot t : t \in T, 0 \leq j \leq 1\}$. Then P is a subgroup of order $3 \cdot 3^3 = 3^4$, an 3-Sylow subgroup of S_9. i.e,

$$P = (\{(1, 2, 3), (1, 4, 7)(2, 5, 8)(3, 6, 9)\})$$

is an 3-Sylow subgroup of S_9. □

4. Adopt the method used in Problem 3 to find generators for a) a 2-Sylow subgroup of S_6. By the result of Problem 2, the order of 2-Sylow subgroup of S_6 is 2^4. We divide $\{1, 2, 3, 4, 5, 6\}$ into 3 clumps:

$$\{1, 2\},\ \{3, 4\},\ \{5, 6\}.$$

Let $\sigma = (1,3)(2,4)$. Set $P_1 = ((1,2))$, $P_2 = ((3,4))$, $P_3 = ((5,6))$. Let $T = P_1P_2P_3$. Note that $\sigma^2 = e, \sigma \notin T$ and $\sigma^{-1}T\sigma = T$ so that T is a subgroup of S_6. Define $P = \{\sigma^j \cdot t : t \in T, 0 \leq j \leq 1\}$. Then P is a subgroup of order $2 \cdot 2^3 = 2^4$, an 2-Sylow subgroup of S_8. i.e,

$$P = (\{(1,2),(5,6),(1,3)(2,4)\})$$

is an 2-Sylow subgroup of S_6.
b) a 3-Sylow subgroup of S_6. By the result of Problem 2, the order of 3-Sylow subgroup of S_6 is 3^2. We divide $\{1,2,3,4,5,6\}$ into 2 clumps:

$$\{1,2,3\},\ \{4,5,6\}$$

Let $\sigma = (4,5,6)$. Set $T = ((1,2,3))$. Note that $\sigma^3 = e, \sigma \notin T$ and $\sigma^{-1}T\sigma = T$. Define $P = \{\sigma^j \cdot t : t \in T, 0 \leq j \leq 1\}$. Then P is a subgroup of order $3 \cdot 3 = 3^2$, an 3-Sylow subgroup of S_6. i.e,

$$P = (\{(1,2,3),(4,5,6)\})$$

is an 3-Sylow subgroup of S_6.
5. If p is a prime number, give explicit generators for a p-Sylow subgroup of S_{p^2}.

Solution. By dividing $\{1,2,\cdots,p^2\}$ into p clumps:

$$\{1,2,\cdots,p\},\ \{p+1,p+2,\cdots,2p\},\cdots,\{p(p-1)+1,p(p-1)+2,\cdots,p^2\},$$

let $\sigma = (1,p+1,\cdots,p(p-1)+1)(2,p+2,\cdots,p(p-1)+2)\cdots(p,2p,\cdots,p^2)$. Set $P_i = (((i-1)p+i,(i-1)p+2i,\cdots,(i-1)p+pi))$, $i = 1,2,\cdots,p$. Let $T = \prod_i P_i$. Set $P = \{\sigma^j \cdot t : t \in T, 0 \leq j \leq p-1\}$. Then P is a subgroup of order p^{1+p}, a p-Sylow subgroup of S_{p^2}. That is,

$$P = (\{\sigma,(1,2,\cdots,p)\})$$

is an p-Sylow subgroup of S_{p^2}. □

6. Discuss the number and nature of the 3-Sylow subgroups and 5-Sylow subgroups of a group of order $3^2 \cdot 5^2$.

Solution. Let H be the 5-Sylow subgroup of given group G of order $3^2 \cdot 5^2$. Note that there are $n_5 = 1+5k$ 5-Sylow subgroup G. Since there is no k satisfying $1+5k \mid 9$, the conjugates of H is itself and hence H is normal in G. Let K be the 3-Sylow subgroup of G. Then there are $n_3 = 1+3k$ 3-Sylow subgroup of G. The possible values of n_3 are: $n_3 = 1, 25$. In particular, if $n_3 = 1$, then K is normal in G. Hence, HK is abelian and $HK = G$, so that G is abelian. □

7. Let G be a group of order 30. a) Show that a 3-Sylow subgroup or a 5-Sylow subgroup of G must be normal in G.

Proof. For the sake of contradiction, assume that both 3-Sylow subgroups and 5-Sylow subgroups of G are not normal in G. Let n_3 and n_5 denote the number of 3-Sylow subgroups and 5-Sylow subgroups of G respectively. Note that $n_3 \in \{1, 10\}$ and $n_5 \in \{1, 6\}$. In our case, $n_3 = 10$ and $n_5 = 6$. $n_3 = 10$ implies that there exists at least 20 distinct elements of order 3 and $n_5 = 6$ implies that there exists at least 24 distinct elements of order 5. Since $o(G) = 30$, this is absurd. Hence, at least one of 3-Sylow subgroup or a 5-Sylow subgroup of G must be normal in G. □

b) From part a) show that every 3-Sylow subgroup and every 5-Sylow subgroup of G must be normal in G.

Proof. It would be better if we prove c) first. Since from a), we know that there exists a 3-Sylow subgroup P_1 or a 5-Sylow subgroup P_2, at least one of them is normal in G. Consequently P_1P_2 is a subgroup of G, with order 15. Since $[G : P_1P_2] = 2$, P_1P_2 is normal in G. Moreover, recall that any group of order 15 is abelian. We introduce a lemma:

Lemma. Let G be a group and N a normal subgroup of G. Suppose a p-Sylow subgroup P of N is normal in N. Then P is also normal in G.
($\Rightarrow$) Since P is normal in N, P is the only p-Sylow subgroup of N. For all $g \in G$, since $gPg^{-1} \subset N$, $o(gPg^{-1}) = o(P)$, $gPg^{-1} = P$. Therefore, P is normal in G.

Note that the group P_1P_2 is normal in G with 3-Sylow subgroup P_1 and 5-Sylow subgroup P_2. As P_1P_2 is abelian, P_1 and P_2 are normal in P_1P_2. Now applying the lemma, we have that P_1 and P_2 are normal in G. Hence proved. □

c) Show that G has a normal subgroup of order 15.

Proof. Refer the above Problem 7 b). □

† **Notification!** Before heading to next problem, we introduce a new concept called "semi-direct product" of a group. Since this is not explicitly handled in the Herstein's text, we make some encounterance with this new concept to deal with other problems in a more delicate manner.

Definition: Let G be a group. Suppose H is a normal subgroup of G and K an arbitrary subgroup of G such that $H \cap K = (e)$ and so that $HK = G$. Then we call G a semi-direct product of H and K.

Lemma. Given groups H and K with a homomorphism given $\phi : K \to \mathscr{A}(H)$, where $k \mapsto \phi_k = T_k \in \mathscr{A}(H), \phi_k(h) = hT_k = khk^{-1}$, there is a semi-direct product group G based on this information and vice-versa.
($\Leftarrow$) Suppose $h, h' \in H$ and $k, k' \in K$. Observe that

$$(hk)(h'k') = h"k"$$

where $h" = h\phi_k(h')$, $k" = kk'$, $(hk)^{-1} = \phi_{k^{-1}}(h^{-1})k^{-1}$. So we have seen that every group with semi-direct product structure depends on the information of homomorphsim $\phi : K \to \mathscr{A}(H)$. ($\Rightarrow$) Conversely, with the information of $\phi : K \to \mathscr{A}(H)$, we can construct a semi-direct product structure on G based on ϕ by:

$$(h, k) \cdot (h', k') = (h\phi_k(h'), kk'),$$

with identity $(1, 1)$, and the inverse given by

$$(h, k)^{-1} = (\phi_{k^{-1}}(h^{-1}), k^{-1})$$

for any $h \in H, k \in K$.

We denote a group G with semi-direct product strucuture by: $G \simeq H \rtimes_\phi K$, where ϕ is the homomorphism K to $\mathscr{A}(H)$ determining the nature of the semi-direct product.

d) From part c) classify all groups of order 30.

Proof. Note that G has a normal subgroup N of order 15. Let K be a 2-Sylow subgroup of G. We can write G as semi-direct product of N and K, that is, $G \simeq H \rtimes_\phi K$. Now we investigate the homomorphism $\phi : K \to \mathscr{A}(N)$. Note that $N \simeq \mathbb{Z}_{15}$ so that $\mathscr{A}(N) \simeq U_{15}$. Since order of the image of $k \neq e \in K$ under ϕ must divide order of U_{15} by 2, $\phi(k)^2 \equiv 1 \pmod{15}$, so that there are at most 4 possibilities of ϕ. Hence there are at most four groups of order 30 upto isomorphism. Observe the following groups of order 30:

$$\mathbb{Z}_{30}, \quad D_{30}, \quad Z_5 \times S_3, \quad Z_3 \times D_{10}.$$

Note that

I in $\mathbb{Z}_{30}$, there are only two elements of order 2,

II in D_{30}, there are 15 elements of order 2,

III in $Z_5 \times S_3$, there are 3 elements of order 2,

IV in $Z_3 \times D_{10}$, there are 5 elements of order 2,

so that 4 of them are non-isomorphic in nature. Hence, we have classified all groups of order 30. □

e) How many different non-isomophic groups of order 30 are there?

Solution. There are 4 different non-isomophic groups of order 30. □

8. If G is a group of order 231, prove that the 11-Sylow subgroup is in the center of G.

Proof. Let H be a 11-Sylow subgroup of G. Then the number of distinct 11-Sylow subgroup n_{11} must be form of $1 + 11k$ and $n_{11} \mid 21$, so the only case is $n_{11} = 1$. Hence H is normal in G. Now consider the homomorphism ϕ of G on $\mathscr{A}(H)$ defined by $g \mapsto T_g \in \mathscr{A}(H)$. Note that $\mathscr{A}(H) \simeq \mathbb{Z}_{10}$, so that $o(\mathscr{A}(H)) = 10$. Since $\gcd(231, 10) = 1$, this group homomorphism must be a trivial homomorphsim. Hence, $T_g = I \in \mathscr{A}(H)$ for all $g \in G$. Thus, $ghg^{-1} = h$ for all $g \in G$, $h \in H$ implying $h \in Z(G)$. □

9. If G is a group of order 385 show that its 11-Sylow subgroup is normal and its 7-Sylow subgroup is in the center of G.

Proof. Let H be a 11-Sylow subgroup of G and n_{11} be the number of distinct 11-Sylow subgroups. Since $n_{11} = 1 + 11k \mid 35$, the only possible case is $n_{11} = 1$ so that H is normal in G. Now let K be a 7-Sylow subgroup of G and n_7 be the number of distinct 7-Sylow subgroups. Similarly above, the only possible case is $n_7 = 1$ so that K is normal in G. Now, consider a homomorphism $\phi : G \to \mathscr{A}(K)$. Since $\mathscr{A}(K) \simeq \mathbb{Z}_6$ and $\gcd(385, 6) = 1$, ϕ is a trivial homomorphism so that $K \subset Z(G)$. □

10. If G is of order 108 show that G has a normal subgroup of order 3^k, where $k \geq 2$.

Proof. Let H be a 3-Sylow subgroup of G. If H is normal in G, then we are done. If not, let S be the set of left cosets of H in G. Considering a left group action of G on S, since $108 \nmid [G : H]! = 4! = 24$, H must contain a non-trivial normal group of G. Since $108/3 = 36 > 24$, such non-trivial group having order 3 is definitely not the case. Thus, G must admit a normal subgroup of order 3^k, $k \geq 2$. □

11. If $o(G) = pq$, p and q are distinct primes, $p < q$, show a) if $p \nmid (q - 1)$, then G is cyclic.

Proof. Let K be a q-Sylow subgroup of G. Then since $pq \nmid p!$, K must be normal in G. Let H be a p-Sylow subgroup of G. We can express G into semi-direct product of K and H, that is, $G \simeq K \rtimes_\phi H$. Here ϕ is a homomorphism H on $\mathscr{A}(K)$. If $p \mid (q-1)$, ϕ must be the trivial homomorphism so that $K \subset N(H)$. Since $H \subset N(H)$ always, $N(H)$ must be a subgroup of G with more than $p + q - 1$ elements, that is, by Lagrange's theorem, $N(H) = pq$, so that H is normal in G. Consequently, G is product of two normal subgroups of orders each relatively prime , hence abelian. Now apply the result of Problem 25 of Section 2.5. Therefore, we can obtain an element of order pq and hence G is cyclic. □

b) If $p \mid (q - 1)$, then there exists a unique non-abelian group of order pq.

Proof. We continue with the discussions above. Suppose $p \mid (q-1)$. Note that then homomorphism $\phi : H \to \mathscr{A}(K)$ must be non-trivial. Since $\mathscr{A}(K)$ is cyclic with order $q-1$, it admits an unique subgroup of order p. As ϕ imbedds H on $\mathscr{A}(K)$, for any such homomorphism ϕ and ψ, $\phi(K) = \psi(K)$. Now these two homomorphism gives isomorphic semi-direct product. Hence, there exists a unique non-abelian group of order pq. $\square$

12. Let G be a group of order pqr, $p < q < r$ primes. Prove a) the r-Sylow subgroup is normal in G.

Proof. Let us denote an r-Sylow subgroup by P_r and let n_r be the number of distinct r-Sylow subgroups in G. Note that the possible values of n_r are 1 and pq. Suppose $n_r = 1$, then P_r is clearly normal in G. If $n_r = pq$, we claim that there is no normal q-Sylow subgroup in G. Let P_q denote an q-Sylow subgroup of G. If P_q is normal G, then the factor group G/P_q has order pr. We know that the r-Sylow subgroup P_r' of G/P_q is normal in its factor group. Hence there is a normal subgroup H of G satisfying $H/P_q \simeq P_r'$ so that $o(H) = qr$. Note that the r-Sylow subgroup of H is normal in H, and since H is normal in G, the r-Sylow subgroup of H is also normal in G. But this also implies that P_r is normal in G, a contradiction. So the q-Sylow subgroup P_q is never normal in G. If we denote n_q as the distinct number of q-Sylow subgroup in G, the possible values of n_q is now r and pr. Suppose $n_q = r$. Note that if $n_r = pq$, there are $pq(r-1)$ distinct elements of order r. Similarly, $n_q = r$ implies there are $r(q-1)$ distinct elements of order q. Note that $pq \leq (q-1)r$ in general. Hence

$$o(G) \geq pqr - pq + (q-1)r \geq pqr,$$

but this implies that there is no room left for elements of order p. This is absurd. Similar method can be applied for the cases $n_q = pr$. Hence, the r-Sylow subgroup must be normal in G. $\square$

b) G has a normal subgroup of order qr.

Proof. From a) we know that r-Sylow subgroup P_r of G is normal. Hence, considering the factor group G/P_r, its order is pq. Now we know that every group of pq, $q > p$ must admit a normal q-Sylow subgroup P_q'. Consequently, there is a normal subgroup H of G such that $H/P_r \simeq P_q'$, where $o(H) = qr$. Therefore, H is the required normal subgroup of order qr. $\square$

c) if $q \nmid (r-1)$, the q-Sylow subgroup of G is normal in G.

Proof. From b) we know that there is a normal subgroup H of qr in G. Let P_q be the q-Sylow subgroup of H. If $q \nmid (r-1)$, P_q must be normal in H. Since H is also normal in G, so does P_q in G. Therefore, q-Sylow subgroup of G must be normal in G. $\square$

13. If G is of order p^2q, p, q primes, prove that G has a nontrivial normal subgroup.

Proof. Let P be the p-Sylow subgroup of G. Note that $p^2 q \nmid [G : P]! = q!$. Hence there is a non-trivial normal subgroup of G contained in P. $\square$

14. If G is of order $p^2 q$, p, q primes, prove that either a p-Sylow subgroup or a q-Sylow subgroup of G must be normal in G.

Proof. Let n_p and n_q denote the number of distinct p-Sylow and q-Sylow subgroups respectivly.

I If $p > q$, then $n_p = 1 + kp \nmid q$, so the only possibility is $n_p = 1$. Hence the p-Sylow subgroup must be normal in G.

II If $p < q$, then $n_q = 1$ or $n_q = p^2$. If $n_q = 1$ was the cases, we are done. If $n_q = p^2$, then there are $p^2(q-1) = p^2 q - p^2$ elements of order q in G.This implies that there must be only one room for p-Sylow subgroup in G, and hence normal.

Therefore, G must have either a normal p-Sylow subgroup or a normal q-Sylow subgroup. $\square$

15. Let G be a finite group in which $(ab)^p = a^p b^p$ for every $a, b \in G$, where p is a prime dividing $o(G)$. Prove
a) The p-Sylow subgroup of G is normal in G.

Proof. Let $S = \{x \in G : x^{p^m} = e$, for some m depending on $x.\}$ Then we know that S is normal in G(by Problem 20 a), Section 2.6.) Let P be the p-Sylow subgroup of G. Clearly $P \subset S$ and P is normal in S. Therefore, P is also normal in G. $\square$

b) If P is the p-Sylow subgroup of G, then there exists a normal subgroup N of G with $P \cap N = (e)$ and $PN = G$.

Proof. Let $o(G) = p^r m$. Consider a homomorphism $\psi : G \to G$ defined by $\psi(a) = a^p$. Note that there is $k \in Z$ such that $p^k \equiv 1 \pmod m$(in particular, take $k = \phi(m)$). Then the kernel of ψ^k is P. Now let N be the image of ψ^k. Since G/P has order m, so does N. Let $a \in N$. Then $a^m = e$. Conversely, if $a^m = e$, $\psi^k(a) = a$ implying that $N = \{a \in G : a^m = e\}$. Clearly, N is normal in G and $P \cap N = (e)$, and hence $PN = G$. $\square$

c) G has a non-trivial center.

Proof. Consider $Z(P)$, the center of p-Sylow subgroup P. Since P is a p-group, $Z(P)$ is non-trivial. Note that $PN = G$, $P \cap N = (e)$. Then for all $x \in Z(P)$, x commutes with elements of the form $np \in G$, where $n \in N, p \in P$ so that $x \in Z(G)$. Thus, $(e) \subsetneq Z(P) \subset Z(G)$ and hence $Z(G)$ is non-trivial subgroup of G. $\square$

16. If G is a finite group and its p-Sylow subgroup P lies in the center of G, prove that there exists a normal subgroup N of G with $P \cap N = (e)$ and $PN = G$.

Proof. Make use of Burnside's transfer Theorem or Schur-Zassenhaus Theorem. □

17. If H is a subgroup of G, recall that $N(H) = \{x \in G : xHx^{-1} = H\}$. If P is a p-Sylow subgroup of G, prove that $N(N(P)) = N(P)$.

Proof. Note that $gPg^{-1} = P$ for every $g \in N(P)$. That is, P is the unique p-Sylow subgroup of $N(P)$. Now, choose $g \in N(N(P))$. From the fact that $P \subset N(P)$,

$$gPg^{-1} \subset gN(P)g{-}1 = N(P)$$

so that $gPg^{-1} = P$. Hence, $N(N(P)) = N(P)$. □

18. Let P be a p-Sylow subgroup of G and suppose a, b are in the center of P. Suppose further that $a = xbx^{-1}$ for some $x \in G$. Prove that there exists a $y \in N(P)$ such that $a = yby^{-1}$.

Proof. We consider the centralizer of a, $C(a)$ in G. Note that $P \subset C(a)$. We also know that $xPx^{-1} \subset C(y)$ since for $p \in P$,

$$(xpx^{-1})a = xpx^{-1}(xbx^{-1}) = xpbx^{-1} = xbpx^{-1} = (xbx^{-1})xpx^{-1} = a(xpx^{-1}).$$

Thus, xPx^{-1} and P are p-Sylow subgroup of $C(a)$. Hence, xPx^{-1} is a conjugate of P in $C(a)$, that is, $xPx^{-1} = zPz^{-1}$ for some $z \in C(a)$. Consequently,

$$(z^{-1}x)P(z^{-1}x)^{-1} \implies y = z^{-1}x \in N(P)$$

so that $yby^{-1} = z^{-1}xbx^{-1}z = z^{-1}az = a$. □

19. Let G be a finite group and suppose that ϕ is an automorphism of G such that ϕ^3 is the identity automorphism. Suppose further that $\phi(x) = x$ implies that $x = e$. Prove that for every prime p which divides $o(G)$, the p-Sylow subgroup is normal in G.

Proof. Note that if a group G admits a fixed point free automorphism of prime order, then G is nilpotent(Thompson). Now consider the subnormal chain from the p-Sylow subgroup to the group itself. Since every p-Sylow groups are characteristic, this Sylow subgroup must be normal in every subgroup in the chain and hence in G. □

20. Let G be the group of $n \times n$ matrices over the integers modulo p, p a prime, which are invertible. Find a p-Sylow subgroup of G.

Solution. First we investigate the order of $GL(n, \mathbb{Z}_p)$. We begin filling the first row of the matrix. Since it must be a non-zero vector, there are $p^n - 1$ possibilities. Now filling the second row, it can be every non-zero vector but the multiple of first row. Hence, there are $q^n - q$ possibilities. For the third row, it can be the same but the linear combination of first and second row. That is, we must remove q^2 choices, so that we are left with $q^n - q^2$ possibilities. Repeating the same process we arrive with:

$$o(GL(n, \mathbb{Z}_p)) = \prod_{i=0}^{n-1} (p^n - p^i).$$

Now we claim that

$$H = \left\{ \begin{pmatrix} 1 & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & 1 & a_{23} & \cdots & a_{2n} \\ 0 & 0 & 1 & \cdots & a_{3n} \\ \vdots & & & & \vdots \\ 0 & \cdots & \cdots & \cdots & 1 \end{pmatrix} : a_{ij} \in \mathbb{Z}_p \right\},$$

set of all upper triangular matrices H_0 with $diag(H_0) = (1, 1, \cdots, 1)$ is a p-Sylow subgroup of $GL(n, \mathbb{Z}_p)$. Since H is finite and product of any two upper triangular with diagonal 1 is always upper triangular with same diagonal components, H forms a subgroup of $GL(n, \mathbb{Z}_p)$. We then calculate the order of H. Recall that the order of p-Sylow subgroup of $GL(n, \mathbb{Z}_p)$ is given by $\displaystyle\prod_{i=0}^{n-1} p^i$. Now choose any elements of H. We can fill its i-th row with p^{i-1} choices, so that total of $p \cdot p^2 \cdots p^{n-1} = \displaystyle\prod_{i=0}^{n-1} p^i$ possibilities. Thus, H is the p-Sylow subgroup of $GL(n, \mathbb{Z}_p)$. □

21. Find the possible number of 11-Sylow subgroups, 7-Sylow subgroups, and 5-Sylow subgroups in a group of order $5^2 \cdot 7 \cdot 11$.

Solution. Let n_5, n_7, n_{11} denote the number of $5, 7, 11$-Sylow subgroups of G respectively. For n_5, it is must that $n_5 = 1 + 5k \mid 77$. But $n_5 = 1$ is the only possible case. Hence, there is only one 5-Sylow subgroup in G. For n_7, $n_7 = 1 + 7k \mid 5^2 \cdot 11$. But $\gcd(274, 7) = 1$ implies that $n_7 = 1$ so that only one 7-Sylow subgroup exists. For n_{11}, $n_{11} = 1 + 11k \mid 5^2 \cdot 7$ but again $n_{11} = 1$ is the only possible case. Thus, every $5, 7, 11$-Sylow subgroups is unique in G. □

22. If G is S_3 and $A = ((1, 2))$ in G, find all the double cosets AxA of A in G.

Solution. Observe that

$$AeA = A = \{e, (1, 2)\}, \quad A(1, 3)A = \{(1, 3), (2, 3), (1, 2, 3), (1, 3, 2)\}.$$

□

23. If G is S_4 and $A = ((1,2,3,4))$, $B = ((1,2))$, find all the double cosets AxB of A, B in G.

Solution. Observe that

$$\begin{aligned}
AeB = AB &= \{id, (1,2,3,4), (1,3)(2,4), (4,3,2,1), (1,2), (2,3,4), (1,3,2,4), (1,4,3)\},\\
A(1,3)B &= \{(1,3), (1,2)(3,4), (2,4), (1,4)(2,3), (1,3,2), (3,4), (1,2,4), (1,4,2,3)\},\\
A(1,4)B &= \{(1,4), (1,2,3), (1,3,4,2), (2,4,3), (1,4,2), (2,3), (1,3,4), (1,2,4,3)\}.
\end{aligned}$$

□

24. If G is the dihedral group of order 18 generated by $a^2 = b^9 = e$, $ab = b^{-1}a$, find the double cosets for H, K in G, where $H = (a)$ and $K = (b^3)$.

Solution. Observe that

$$\begin{aligned}
HeK = HK &= \{id, a, b^3, b^6, ab^3, ab^6\},\\
HbK &= \{b, ab, b^4, b^7, ab^4, ab^7\},\\
H(b^2)K &= \{b^2, ab^2, b^5, b^8, ab^5, ab^8\}.
\end{aligned}$$

□