Topics in Algebra solution

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Problems in Section 3.5.

1. Let R be a ring with unit element, R not necessarily commutative, such that the only right-ideals of R are (0) and R. Prove that R is a division ring.

Proof. If $R = (0)$, then there is nothing to prove. So we assume that $R \neq (0)$. Choose $a \neq 0 \in R$. Then aR is a nontrivial right ideal of R so that $aR = R$. Since R has the unit element 1, there is $b \in R$ such that $ab = 1$. Hence, R is a division ring. □

2. Let R be a ring such that the only right ideals of R are (0) and R. Prove that either R is a division ring or that R is a ring with a prime number of elements in which $ab = 0$ for every $a, b, \in R$.

Proof. If $R = (0)$, there is nothing to prove. Assume that $R \neq (0)$. Now consider the following subset U of R

$$U = \{x \in R : ax = 0 \text{ for all } a \in R\}$$

Note that U is a right-ideal of R. Hence, it it either $U = (0)$ or $U = R$. Suppose $U = R$. Then $ab = 0$ for all $a, b \in R$. Suppose V is an addtive subgroup of R. Since product of any elements of R is 0 and $0 \in V$, V is also a (right) ideal of R. That is, R admits no nontrivial proper subgroup. Hence, R must be of an additive subgroup of prime order.
Now we assume that $U = (0)$. Then, for each $a \neq 0 \in R$, there is $r \in R$ such that $ar \neq 0$. We will establish the existence of (right) multiplcative identity and inverse in R. Now consider the right ideal aR. Then $aR \neq (0)$ so that $aR = R$. Now for some $u \in R$, $au = a$. We claim that u is the required multiplicative identity(unit element). Note that if $xy = 0$, $x = 0$ or $y = 0$ in R otherwise if $x, y \neq 0$, since $xR = R$, $yR = R$, $0 = (xy)R = x(yR) = xR = R$ which not the case. Now from $au = a$, $auu = au \iff a(u^2 - u) = 0$ so that $u^2 = u$. Now take any $b \in R$. Then $bu = bu^2 \iff (b - bu)u = 0$ so that $b = bu$ as $u \neq 0$. Thus, u is the right identity of R. Now from $aR = R$ for each $a \neq 0$, $ak = u$ for some $k \in R$. Hence, k plays the role of right inverse. Therefore, R is now a division ring. □

3. Let J be the ring of integers, p a prime number, and (p) the ideal of J consisting of all multiples of p. Prove
a) $J/(p)$ is isomorphic to J_p, the ring of integers mod p.

Proof. Define a mapping $\phi : J_p \to J/(p)$ by $\phi(a) = a+(p)$. This is clearly a homomorphism and also onto. The kernel of ϕ consists of elements a such that $a \in (p)$ where $a = 0$ in J_p. Hence, ϕ has trivial kernel and so that ϕ gives onto isomorphism between J_p and $J/(p)$. $\square$

b) Using Theorem 3.5.1. and part (a) of this problem, that J_p is a field.

Proof. Note that (p) is a maximal ideal in J if and only if p is prime. Hence, $J/(p)$ is a field and since $J/(p) \simeq J_p$, J_p is a field. $\square$

4. Let R be the ring of all real-valued continuous functions on the closed unit interval. If M is a maximal ideal of R, prove that there exists a real number γ, $0 \leq \gamma \leq 1$, such that $M = M_\gamma = \{f(x) \in R : f(\gamma) = 0\}$.

Proof. Note that $[0,1]$ is compact in $\mathbb{R}$. We assume that $M \neq (0)$ as an ideal of R. Further that $M \subsetneq M_\gamma$ for all $\gamma \in [0,1]$. Consequently,

$$[0,1] \subset \bigcup_{f \in M} f^{-1}\left(\mathbb{R} - \{0\}\right)$$

so that $\bigcup_{f \in M} f^{-1}\left(\mathbb{R} - \{0\}\right)$ is an open covering of $[0,1]$. Hence, it has a finite subcover that

$$[0,1] \subset \bigcup_{i=1}^{n} f_i^{-1}\left(\mathbb{R} - \{0\}\right),$$

such that $f_i(x_i) \neq 0$ for each $x_i \in [0,1]$. Now define a function $g : [0,1] \to \mathbb{R}$ by

$$g(x) = \sum_{i=1}^{n} f_i^2(x).$$

Then $g(x) > 0$ for all $x \in [0,1]$ and $g \in M$. Note that $1/g \in R$, so that $1 = 1/g \cdot g \in M$ and hence $M = R$. But since $M_\gamma \neq R$, it is must that $M \subset M_\gamma$ for some $\gamma \in [0,1]$. Suppose M was maximal, $M_\gamma = M$. Hence proved. $\square$